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MATHEMATICAL MODELING OF FLOW ALONG A CYLINDER IN A HOMOGENEOUS GRANULAR BED

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A network model of flow along a cylinder placed in a stationary porous medium with uniform permeability is proposed. The results of numerical modeling for different regimes of liquid flow are given.

The problem of flow along impermeable surfaces in contact with an infiltrated granular bed arises in the study of heat and mass transfer processes in a catalyst bed in chemical apparatuses [1], in heat-releasing elements of nuclear reactors [2], etc. The pore space inside these beds forms a system of communicating curvilinear channels of variable cross section, the character of flow in which is governed by local geometric characteristics of the fill, the properties of the flowing medium, and the regime parameters. A mathematical description of the processes in a granular bed in solving applied problems is constructed with the use of a porous body model developed within the framework of the filtration theory.

The study of hydraulic characteristics in packings of different structures has been the objective of numerous experimental and theoretical works [1-8 and others]. However, methodological difficulties arising in carrying out physical experiments do not yet permit the formulation in ultimate form of mathematical models that would fairly completely describe real processes. In this connection, the problem of establishing numerical methods that permit the study of hydrodynamic and thermal processes in granular beds on the basis of numerical experiment remains topical.

Statement of the Problem. We consider a two-dimensional model of isothermal motion of a liquid in the vicinity of an impermeable cylinder placed in a porous medium. We represent the system of equations that describes this motion as:

$$-\operatorname{grad} p = a\mathrm{U} + b\,|\mathrm{U}|\,\mathrm{U}\,; \tag{1}$$

$$\operatorname{div}(\rho \mathrm{U}) = 0\,; \tag{2}$$

$$xu + yv = 0 \quad \text{for} \quad x^2 + y^2 = R^2\,; \tag{3}$$

$$\mathrm{U} = (u\,,\ v) \to (\vec{U_\infty}, \vec{0}) \quad \text{as} \quad x^2 + y^2 \to \infty\,. \tag{4}$$

Here p is the pressure; U is the liquid velocity; R is the radius of the cylinder in the flow. The coefficients a and b, in accordance with the flow regime, were computed on the basis of experimental data obtained in [2, 4].

We are able to obtain, in general form, an analytical solution to the problem of nonlinear filtration for a certain class of nonlinear equations and some particular statements of the problem [6, 7, and others]. Thus, for example, in [7] a special nonlinear law of filtration is introduced for which the basic system of equations obtained by transformation of the hodograph has a general solution that enables us to efficiently apply the apparatus of the theory of complex variable functions. Solutions are given for particular problems of steady-state filtration of an incompressible liquid in a band with impermeable boundaries.

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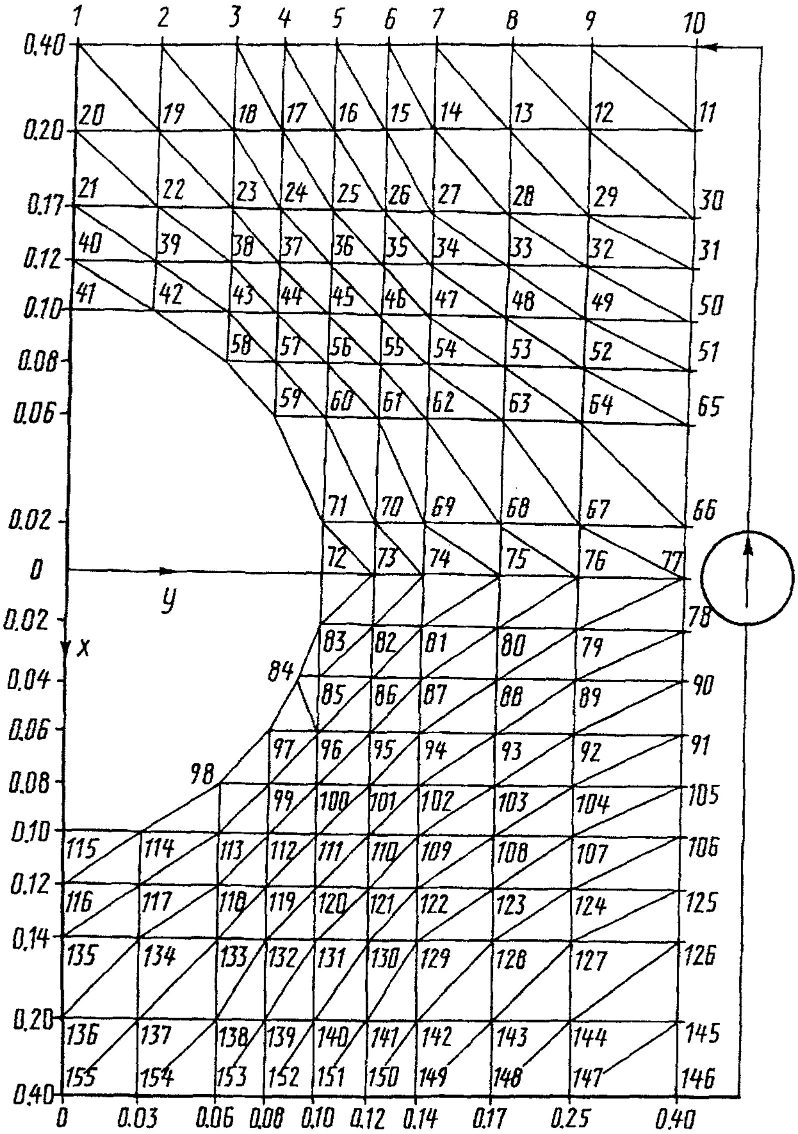

Fig. 1. Network model of flow along a cylinder in a granular bed. x, y, m.

We note that constructing an approximate analytical solution to the problem (1)-(4) as $\mathrm{Re} \to \infty$ ($\mathrm{Re} = Ud\rho/\mu$) leads to great mathematical difficulties. Therefore, the stated problem is solved numerically on the basis of a network method [9, 10] which is an analog of calculation methods for electric and hydraulic circuits. Owing to the use of graph theory, high computational efficiency of these methods is attained.

Numerical Method. To calculate the flow distribution in a network (circuit), contour and nodal methods are used. As applied to a continuous medium, the nodal method is an analog of the finite-difference and finite-element methods. The system of equations in the nodal method is written for pressures in network nodes. In the contour method, pressure is excluded and the system of equations is written for flow rates on portions of the network. The flow rates G are subdivided into two families: G_{ch} on the chords and G_{tr} on the graph tree. The flow rates G_{tr} are expressed using matrix relations of network theory on the basis of a continuity equation in terms of the flow rates G_{ch}. As a result, a final system of nonlinear algebraic equations is obtained for only the flow rates G_{ch}. In solving nonlinear problems, the contour method is more efficient as compared with the nodal one, since it provides a better convergence of the solution [11]. In this connection, the contour method was used in these calculations.

A discrete grid applied in the finite-difference or finite-element methods can serve as a network. By "grid" we mean here a set of sections (branches of the graph) that link the calculated nodes. The sections of the grid are simulated by sections of a circuit with lumped resistances, which are determined by the effective cross-sections of

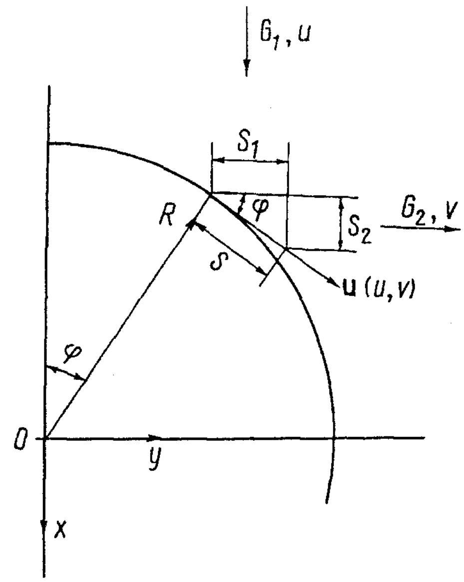

Fig. 2. Design scheme of flow along a curvilinear surface.

the sections S_e. We calculate flow rates of the liquid on the sections of the network and the pressure at the nodes. To change from a conventional grid to a network, it is supplemented by a sections with a pump (in Fig. 1, section 146-10). The constructed hydraulic circuit is an oriented graph. The directions of the sections of the graph are prescribed in an arbitrary way. In Fig. 1, the calculated region is "covered" with an Eulerian graph, which permits us to prescribe the graph tree and to automatically construct a system of independent contours [9]. On the sections between nodes 1-10 and 146-155, zero hydraulic resistance is prescribed, which makes it possible to simulate the inlet and outlet collectors. The velocity distribution at the inlet to the calculated region will be governed by the flow of the liquid from infinity with the velocity U_∞. The average velocity at the inlet to the calculated region is calculated from the prescribed Re number. In terms of this velocity we determine the flow rate that is prescribed on section 146-10.

We rewrite Eq. (1) for an arbitrary section of the network with the direction 1 as:

$$- \operatorname{grad} p_l = A_l \, |\mathbf{U}_l| \, \mathbf{U}_l , \tag{5}$$

where

$$A_l = \left(\frac{a}{|\mathbf{U}|} + b \right) \frac{|\mathbf{U}|}{|\mathbf{U}_l|} .$$

We write Eq. (5) in terms of the flow rate G_l in the section in question ($G_l = \rho S_e U_l$):

$$\Delta p_l = R_l \, |G_l| \, G_l , \tag{6}$$

where $R_l = A_l l / (\rho S_e)^2$ is the resistance of the section and l is its length.

We write the continuity equation (2) for an arbitrary node in terms of the flow rates converging in it:

$$\sum_i G_i = 0 . \tag{7}$$

Expressing the flow rates G_{tr} from system of Eqs. (7) in terms of the flow rates G_{ch} and substituting them into the system of equations for the independent contours, we obtain a nonlinear system of algebraic equations for the flow rates G_{ch}, which we solve by Newton's method. We determine the increments in G_{ch} in the n-th iteration from linearized system of equations [10]:

$$W(\mathbf{G}^{(n)}) \, \Delta \mathbf{G}_x^{(n+1)} = - \mathbf{F}(\mathbf{G}^{(n)}) , \tag{8}$$

TABLE 1. Ratio U_{non}/U_{lin} in the Midship Section ($x = 0$)

Re	y, m					
	0.10	0.12	0.14	0.17	0.25	0.40
50	0.95	0.98	1.01	1.04	1.08	1.12
200	0.92	0.96	1.00	1.05	1.09	1.13
1000	0.91	0.96	0.99	1.04	1.09	1.14
500	0.91	0.96	1.00	1.04	1.09	1.13

where $W = 2BR|G|B^T$ is a Jacobi matrix; $F = BR'|G|G$ is the vector of the discrepancies of the heads. Here B is the matrix of the contours; $|G|$ is the diagonal matrix of the absolute values of the flow rates in the sections; R' is the diagonal matrix of resistances of the sections.

The pressure field is found upon completion of the iteration process for the velocities in terms of the computed pressure differences in the sections of the network and the prescribed pressure at the basis node (usually, this is a node at the inlet to the network). In [9], satisfactory agreement of the results of numerical solutions to a nonlinear filtration problem with experiment is reported.

The effective cross-sections of the sections are analogs of the cross-sections of conductors with current. For right triangles, the effective cross-section of the hypotenuse is equal to zero [10], i.e., these sections are of infinitely large resistance and hence the flow rate in them is equal to zero. When a flat plate is in a flow this condition is equivalent to the adhesion condition. We refer to the case of flow along a curvilinear surface (Fig. 2). Let the flow rate $G_1 = \rho S_1 u = \rho S u \cos\varphi$ arrive in sections S of this surface, whereas the flow rate $G_2 = \rho S_2 v = \rho S v \sin\varphi$ is that reflected from this surface. Since the flow rate in sections S is equal to zero, flow rates G_1 and G_2 are identical and this condition is equivalent to condition (3) (the normal-to-the surface velocity $U_n = 0$). If the liquid is ideal at points on a stationary surface in a flow, the velocity $U(u, v)$ is tangent to the surface ($U_n = 0$). At small filtration rates (the Darcy regime), the flow is potential and this condition can be considered as justified, although it is clear that the resistance force, even in this case, is not equal to zero but is small enough as compared with the resistance of the granular bed. At reasonably large Re, as the calculations show, the no-flow condition (3) results in flow differing insignificantly from a potential one in this case too. At the same time, it is clear that the adhesion condition in filtration flow is also unacceptable [3]. In real flow, part of the flow reflects from the surface, part of the flow slips at a tangent, i.e., condition (3) is not satisfied. Therefore, for a more adequate correspondence between model and natural flows on the sections on the surface of a body in flow, Eq. (1) should be also dealt with (as a boundary condition) [10]. The resistances of these sections depend on the wall porosity and the wall layer thickness.

Results of a Computational Experiment. Numerical investigations of flow along a cylinder in a granular bed are performed with the use of a design graph scheme (see Fig. 1). A homogeneous ball fill (the ball diameter $d = 0.01$ m, the porosity in the layer $\varepsilon = 0.4$) served as a granular bed. The cylinder radius is $R = 0.1$ m; the dimensions of the calculated region are 0.4×0.8 m. The calculated region was divided nonuniformly by linear triangles (smaller calculated cells were used in the vicinity of the cylinder in the flow). As an infiltrated liquid, air at atmospheric pressure and $t = 20°C$ was taken.

The calculations are performed over a wide range of Re numbers: 2 (Darcy flow); 50 (the transient regime – Forchheimer flow [4]); 200; 1000; 5000 (turbulent flow). For each Re number, we calculated flow with the use of empirical dependences of [2] and [4] and wrote the analytical solution of the linear problem at the center of each calculated section:

$$u = U_\infty \, [1 - R^2 (x^2 - y^2)/(x^2 + y^2)^2 \,]; \tag{9}$$

$$v = -\, 2R^2 \, U_\infty \, xy/(x^2 + y^2)^2 ; \tag{10}$$

$$p = p_0 - U_\infty \, [1 + R^2/(x^2 + y^2)\,]\, x\mu/K\,, \tag{11}$$

where K is the permeability of the ball fill; P_0 is the pressure in the basis node (node 10 in Fig. 1).

The results of the calculations performed show that in the considered range of Re numbers, the velocities U_{non} obtained on the basis of the solution to problem (1)-(4) are similar to U_{lin} values (9) and (10) (see Table 1). These results are obtained for the case of the same resistance in the vicinity of the boundary of the cylinder and in the granular layer (ε = const). Analogous results are obtained in [8] for an isotropic and reasonably large coefficient of resistance in the porous body.

An increase in resistance at the boundary of the cylinder leads to an even greater convergence of the solutions to the linear and nonlinear problems (u, v). In the vicinity of surfaces in flow, porosity is higher as a rule than in the fill itself, which leads, on the whole, to a decrease in hydraulic resistance in this region. The influence of the porosity in the wall layer on the hydrodynamics in the fill is dealt with in the case of turbulent flow (Re = 1000). According to [1, 12], the porosity in the vicinity of the wall is taken equal to 0.8; the wall layer thickness is $0.2d$. In this case in the wall layer $U_{non}/U_{lin} \sim 5$, the total pressure drop in the granular layer decreasing by 5%. The perturbation of velocity due to increased porosity near the wall decays exponentially away from the cylinder. This result demonstrates that, in addition to abnormal heat conduction, the wall zone has a no less substantial abnormal hydrodynamics too. It is pertinent to note that as far as the influence on heat transfer is concerned these effects can partially compensate one another, though this calls for a separate additional investigation.

The velocity fields calculated with the use of data of [2] and [4] are in good agreement. At the same time the data of [2] yield a somewhat undervalued pressure drop in the granular layer with respect to [4].

The results obtained permit consideration of the possibility of finding the initial nonlinear problem (1)-(4) by solving two nonlinear problems. Let u_0 and v_0 be the solution of the Darcy linear problem (9) and (10) and p_0 be the pressure calculated from the nonlinear filtration problem (1) in terms of u_0 and v_0. We will seek the solution of problem (1)-(4) as:

$$u = u_0 + u_1\,; \tag{12}$$

$$v = v_0 + v_1\,; \tag{13}$$

$$p = p_0 + p_1\,. \tag{14}$$

We substitute (12)-(14) into (1)-(2). Assuming that corrections v_1, v_1, and p_1 are rather small and confining ourselves to terms of the 1st order of smallness in the expansion of the right side of Eq. (1), we obtain the following linear system for the corrections:

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = 0\,; \tag{15}$$

$$-\frac{\partial p_1}{\partial x} = [a + c\,(2u_0^2 + v_0^2)\,]\, u_1 + c u_0 v_0 v_1\,; \tag{16}$$

$$-\frac{\partial p_1}{\partial y} = c u_0 v_0 u_1 + [a + c\,(u_0^2 + 2v_0^2)\,]\, v_1\,, \tag{17}$$

where

$$c = b/(u_0^2 + v_0^2)^{1/2}\,.$$

Boundary condition (3) for the system of Eqs. (15)-(17) remains as before, and in place of boundary condition (4), a root-mean-square (active) [11] pump head equal to the discrepancy of the heads calculated by Eq. (1) in terms of the linear velocities (9) and (10) is prescribed on each contour.

The obtained linear system of Eqs. (15)-(17) can be used for numerical solution of the nonlinear problem by an iteration process, the velocities, as an initial approximation, being taken from the solution of the Darcy problem and the pressure from the nonlinear filtration Eq. (1) for the calculated velocities. The subsequent approximations are taken according to relations (12)-(14). The iteration process is repeated until the discrepancies of the heads become smaller than the value prescribed.

CONCLUSIONS

1. Based on a network method, a numerical algorithm is developed for modeling flow along a cylinder in a granular bed.

2. A flow field for filtration (nonlinear) flow along a cylinder in a porous medium is constructed using a contour mesh-analysis method. In the case of homogeneous isotropic porosity, numerical results show that the deviation of the velocity field from the linear case (the Darcy law) does not exceed 10–12% for Re as large as 5000. At the same time if there is a thin layer of increased porosity near the body in the flow, the calculations show a substantial difference of the nonlinear hydrodynamics from linear one. For Re = 1000, the nonlinear velocity in the wall zone has a five-fold excess over the linear velocity.

3. A system of linear equations for calculating nonlinear corrections on the basis of linearization of the initial nonlinear filtration equation is obtained.

NOTATION

x, y, rectangular coordinates; R, radius of cylinder; d, diameter of ball; ρ, density; μ, dynamic viscosity; K, permeability; p, pressure; u, v, components of velocity vector U; G, flow rate of liquid.

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